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TANGENT DEVELOPABLE SURFACES ELEMENTS IN THIN-WALLED STRUCTURES

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Summary. This report demonstrates the capabilities of an advanced research area of the applied mathematics, i.e., computational geometry to be applied for shaping dimensional structures. Vector-matrix models are provided to purpose of piecewise-smooth structures modelling by surface elements with zero Gaussian curvature (elements of developable surfaces). The elements of tangent developable surfaces can be built on the directing curves pieces located arbitrarily in space.

The paper presents an analytical algorithm for drawing a cutting for the tangent developable surface element with two specified directing curves and the edge of regression known. An analytic algorithm for the curve on tangent developable surface development via the parametric equations of the edge of regression and the curve itself is obtained based on the tangent developable surface edge of regression development algorithm.

1 INTRODUCTION

Developable surfaces can be used as structure elements and find wide application in construction, engineering and textile industries [1, 2]. Developable surfaces are especially in demand in tent and plate structures, as they provide the possibility of making structures from flat pieces with subsequent connections along the cut lines.

Using of developable surface elements in the design and manufacture of sheet and textile structures one can apply analytical algorithms for modeling surface features and their developments.

The use of analytical algorithms for the elements of cylindrical and conical surfaces development in the design of tent structures is demonstrated in [3]. Tangent developable surfaces are deployable surfaces as well and can be used in the design of the tent and metal sheet structures along with cylindrical and conical surfaces, and in some cases can provide wider choice of surfaces forms.

2 METHODS OF TANGENT DEVELOPADLE SURFACE ELEMENTS MODELLING

One of the ruled surfaces, which can be used in geometric modeling, is tangent developable surfaces – the surface of tangents to an arbitrary smooth space curve. The tangent developable surface at every point has zero Gaussian curvature, and according to the differential geometry methods is a developable ruled surface, which means that an element of such surface can be unfolded into a plane without stretching or tearing. In the fabrication of different plate, tent and others structures this property allows to use materials which are not amenable to stretching.

The general equation of the tangent developable surface, based on its definition as a surface of tangents to a space curve, can be written in the form:

$$\hat{r}(u,v)=\hat{r}_d(u)+v\hat{\tau}(u), \tag{1}$$

where $\hat{r}_d(u)$ = a directive curve, $\hat{\tau}$ = unit tangent vector to the directive curve defined by the relationship

$$\hat{\tau}(u)=\frac{\dfrac{d\hat{r}_d}{du}}{\left|\dfrac{d\hat{r}_d}{du}\right|}. \tag{2}$$

Using relation (2) the general equation of the tangent developable surface can be written in the form^

$$\hat{r}(u,v)=\hat{r}_d(u)+v\frac{d\hat{r}_d/du}{\left|d\hat{r}_d/du\right|}. \tag{3}$$

The examples of tangent developable surface elements created with this method are presented in Figures 1 and 2.

a b

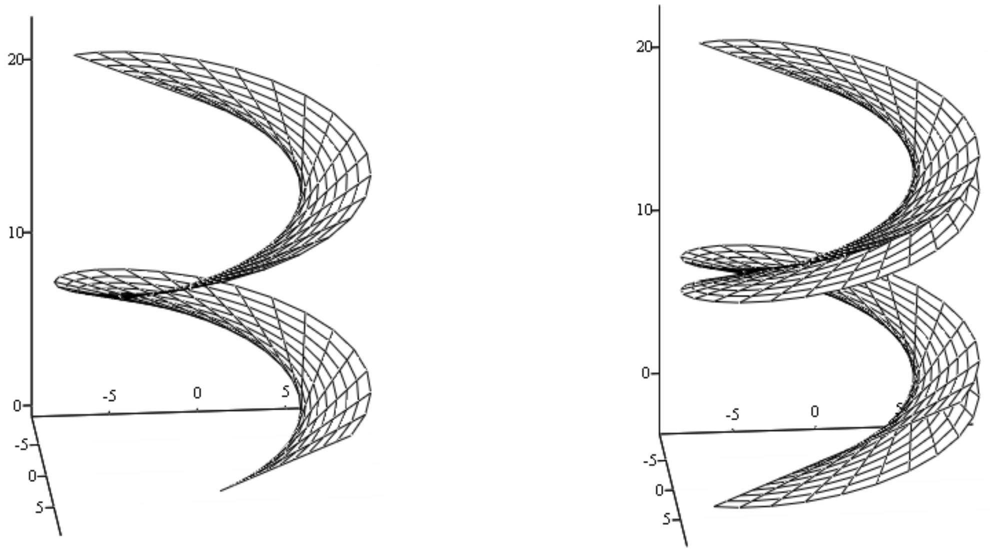

Figure 1. Tangent developable surface with directive curve in form of the circular helix $\vec{r}_d(u)=\{5\cos u, 5\sin u, u\}$, $0\le u\le 3\pi$:
a – with $0\le v\le 8$; b – with $-8\le v\le 8$

Figure 1*a* shows the tangent developable surface with the directive curve in the form of circular helix when parameter ranging $0 \le v \le 8$. This surface is referred to as the Archimedes screw and was used in ancient times in the simplest hydraulic structures. Figure 1*b* shows the same tangent developable surface with parameter ranging $-8 \le v \le 8$, and this illustrates the directive curve to be the edge of regression of tangent developable surface and that the surface itself consists of two cavities. The tangent developable surface with the directive curve in form of helical spiral is shown in the Figure 2.

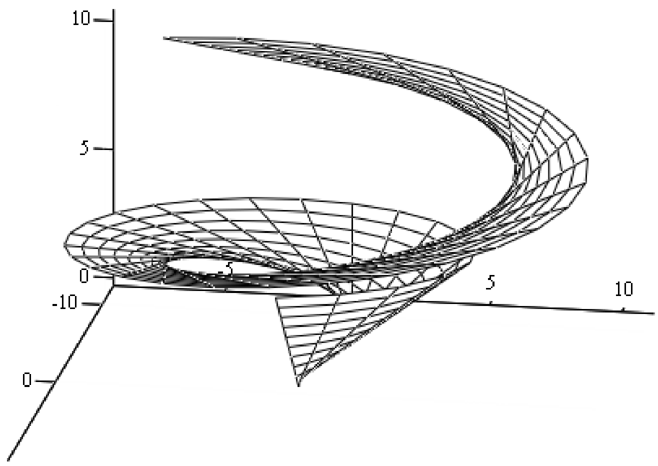

Figure 2. Tangent developable surface with directive curve in form of the helical spiral

$\vec{r}_d(u) = \{u\cos u; u\sin u; u\}$, $0 \le u \le 3\pi$, $0 \le v \le 7$

Another method of tangent developable surface modeling is to build surface elements on the segments of the directive curves located arbitrarily in space. A large number of options for modelling of tangent developable surface elements with different directive curves are given in the *Encyclopedia of analytical surfaces*[4]. Using one of the algorithms suggested in *Encyclopedia of analytical surfaces*[4], we can make a model of a tangent developable surface element on the ellipse and the parabola, which are located in parallel planes.

Let us define the ellipse by the matrix equation

$$\hat{r}_1(u) = \begin{pmatrix} a\cos u \\ 0 \\ b\sin u \end{pmatrix}, \ u_1 \le u \le u_2, \tag{4}$$

and the parabola by the matrix equation

$$\hat{r}_2(v)=\begin{pmatrix} 2cv \\ l \\ \left(b-cv^2\right) \end{pmatrix},\ v_1 \le v \le v_2. \tag{5}$$

Under the uniqueness of the tangent developable surface containing an ellipse and a parabola located in parallel planes, it is possible to find the relation between the parameters of directive curves $v=(b/a)\tan u$. Then, the equation of the tangent developable surface element will be described by the formula:

$$\hat{r}(u,k)=k\begin{pmatrix} a\sin u \\ 0 \\ b\cos u \end{pmatrix}+(1-k)\begin{pmatrix} 2c\dfrac{b}{a}\tan u \\ l \\ \left(b-c\left(\dfrac{b}{a}\tan u\right)^2\right) \end{pmatrix},\ u_1 \le u \le u_2,\ 0 \le k \le 1. \tag{6}$$

This tangent developable surface element with parameters $a=2$, $b=3$, $c=0.4$, $l=4$, $-1 \le u \le 1$, $0 \le k \le 1$ is shown in Figure 3.

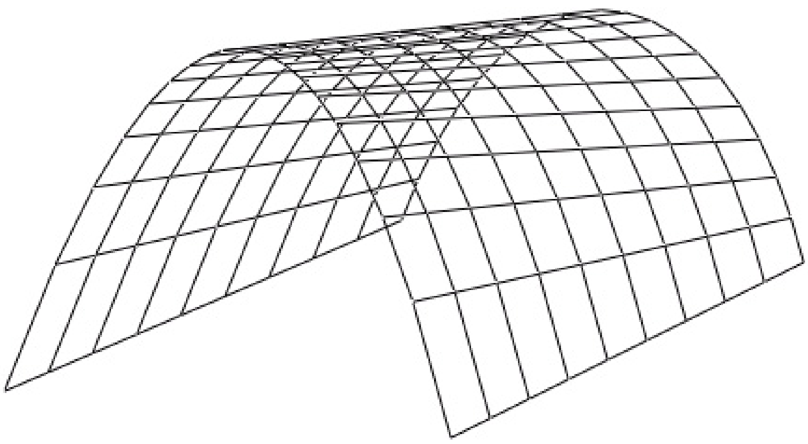

Figure 3. Tangent developable surface with ellipse and parabolic curve for directive lines and parameters within ranges $-1 \le u \le 1$, $0 \le k \le 1$

Another example is the tangent developable surface element formed on both the circumference and the parabola which are located in parallel planes and set by the corresponding equations:

$$\hat{r}_1(u)=\begin{pmatrix} -\sqrt{R^2-u^2} \\ 0 \\ u \end{pmatrix},\ u_1 \le u \le u_2 \text{ and } \hat{r}_2(v)=\begin{pmatrix} \dfrac{v^2}{2p} \\ l \\ v \end{pmatrix},\ v_1 \le v \le v_2. \tag{7}$$

In this case, the equation of the tangent developable surface element takes the following form[4]:

$$\hat{r}(\lambda,\beta)=\begin{pmatrix} (\lambda-1)\sqrt{R^2-\beta^2}+\dfrac{\lambda p\beta^2}{2\left(R^2-\beta^2\right)} \\ \lambda l \\ \beta\left(1-\lambda+\dfrac{p\lambda}{\sqrt{R^2-\beta^2}}\right) \end{pmatrix},\ 0 \le \lambda \le 1, \tag{8}$$

and the equation of the edge of regression to this surface can be written as

$$\hat{r}_d(\beta)=\begin{pmatrix} \dfrac{h}{1-\dfrac{pR^2}{\left(R^2-\beta^2\right)^{\frac{3}{2}}}} \\ \dfrac{p}{2}\sqrt{R^2-\beta^2}\left[\dfrac{\beta^2+2R^2}{\left(R^2-\beta^2\right)^{\frac{3}{2}}-pR^2}\right] \\ -\dfrac{p\beta^3}{\left(R^2-\beta^2\right)^{\frac{3}{2}}-pR^2} \end{pmatrix}. \tag{9}$$

The tangent developable surface element with parameters $R=4$, $p=2$, $l=6$, $-3.5 \le u \le 3.5$, $-3.5 \le v \le 3.5$ is presented in Figure 4.

Another way to model a tangent developable surface element is to write the equation of the ruled surface with two directive curves in the following form:

$$\hat{r}(u,v)=v\hat{r}_d(u)+(1-v)\hat{r}_1(u),\ 0 \le v \le 1,\ u_1 \le u \le u_2, \tag{10}$$

provided that the first guiding line is the edge of regression for the considered tangent developable surface and the second guiding line is expressed through the edge of regression:

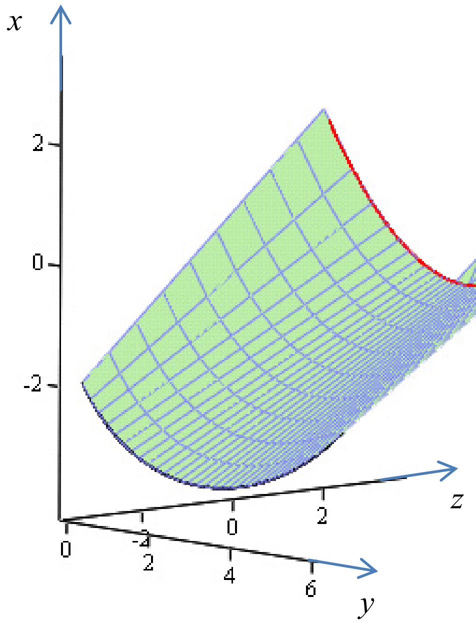

Figure 4. Tangent developable surface with directive lines in the form of circle and parabolic curves, parameters within ranges $-3.5 \le u \le 3.5$, $-3.5 \le v \le 3.5$

$$\hat{r}_1(u) = \hat{r}_d(u) + v(u)\hat{\tau}(u). \tag{11}$$

Here $\hat{\tau}$ is the unit tangent vector to the edge of regression $\hat{r}_d(u)$.

If we set the edge of regression and the second guiding line by the following equations

$$\hat{r}_d(u) = \begin{pmatrix} \cos u \\ \sin u \\ u \end{pmatrix}, \; u_1 \le u \le u_2, \tag{12}$$

$$\hat{r}_1(u) = \hat{r}_d(u) + \sqrt{2}\hat{\tau}, \; u_1 \le u \le u_2 \tag{13}$$

and perform the necessary transformations, the equation of the tangent developable surface element (Figure 5) takes the form:

$$\hat{r}(u,v) = v\hat{r}_d(u) + (1-v)\hat{r}_1(u) = v\begin{pmatrix}\cos u\\ \sin u\\ u\end{pmatrix} + (1-v)\begin{pmatrix}\cos u - \sin u\\ \sin u + \cos u\\ u+1\end{pmatrix}, \tag{14}$$

$$0 \le v \le 1,\ u_1 \le u \le u_2.$$

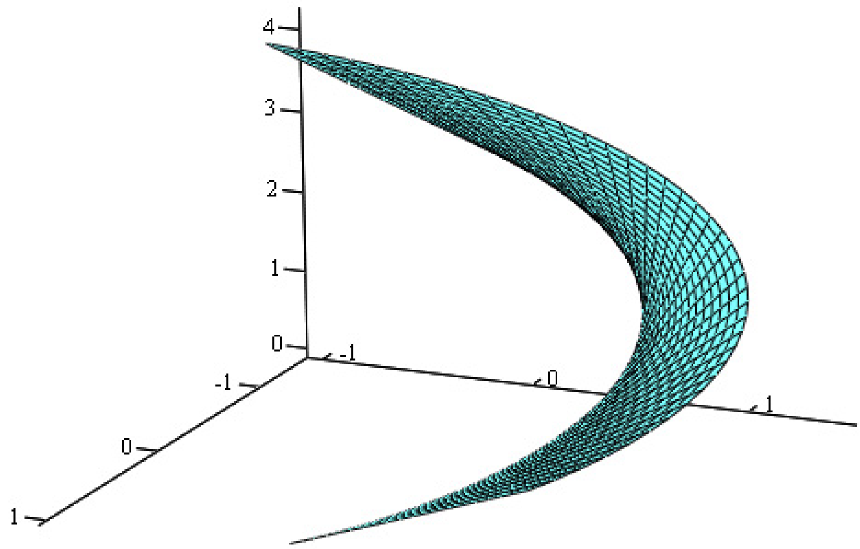

Figure 5. Tangent developable surface with directing lines $\hat{r}_d(u) = \begin{pmatrix}\cos u\\ \sin u\\ u\end{pmatrix}$ and $\hat{r}_1(u) = \hat{r}_1(u) + \sqrt{2}\hat{\tau}$ and parameter $0 \le u \le \pi$

3 ANALYTICAL DEVELOPING METHODS OF THE CURVE LINES BELONING TO TANGENT DEVELOPABLE SURFACES

Based on the condition of the lengths of the arcs *ds* and the curvature *k* for the edge of regression and its development line in all corresponding points coincide the development of the edge of regression $\vec{r}_d(u)$ of a tangent developable surface can be found. Using the definitions of the planar and spatial curves curvature

$$k = \frac{d\phi}{ds} = \frac{\left|\dot{\vec{r}}_d(u) \times \ddot{\vec{r}}_d(u)\right|}{\left|\dot{\vec{r}}_d(u)\right|^3} \tag{15}$$

we can find the increment of the angle φ when moving a point at a distance *d*s along a planar curve on the plain of development:

$$d\varphi = \frac{\left|\dot{\vec{r}}_d(u) \times \ddot{\vec{r}}_d(u)\right|}{\left|\dot{\vec{r}}_d(u)\right|^3} ds = \frac{\left|\dot{\vec{r}}_d(u) \times \ddot{\vec{r}}_d(u)\right|}{\left|\dot{\vec{r}}_d(u)\right|^2} du\,. \tag{16}$$

If we select the plane of development Oξη, in which the axis Oξ is directed along a tangent to the directive curve in its initial point, the equation of the development curve for the edge of regression takes the following form[5]:

$$\begin{cases} \xi = \int\limits_{u_1}^{u} \left|\dot{\vec{r}}_d(t)\right| \cos\left(\int\limits_{u_1}^{t} \frac{\left|\dot{\vec{r}}_d(u) \times \ddot{\vec{r}}_d(u)\right|}{\left|\dot{\vec{r}}_d(u)\right|^2} du \right) dt, \\ \eta = \int\limits_{u_1}^{u} \left|\dot{\vec{r}}_d(t)\right| \sin\left(\int\limits_{u_1}^{t} \frac{\left|\dot{\vec{r}}_d(u) \times \ddot{\vec{r}}_d(u)\right|}{\left|\dot{\vec{r}}_d(u)\right|^2} du \right) dt, \end{cases} \quad u_1 \le u \le u_2\,. \tag{17}$$

As an example, let us build the development curve for the edge of regression which is defined by the equation (12) and shown in Figure 6.

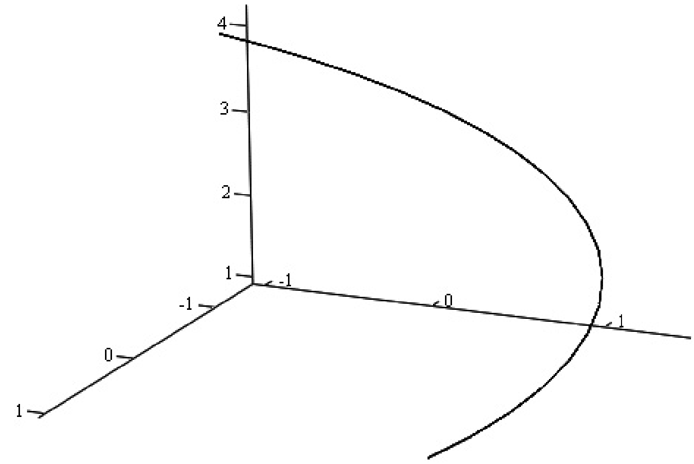

Figure 6. The edge of regression $\hat{r}_d(u) = \begin{pmatrix} \cos u \\ \sin u \\ u \end{pmatrix}$ with parameter $0 \le u \le \pi$

Let us perform the intermediate calculations for finding the development curve equations:

$$\frac{d\hat{r}_d(u)}{du}=\begin{pmatrix}-\sin u\\ \cos u\\ 1\end{pmatrix},\ \frac{d^2\hat{r}_d(u)}{du^2}=\begin{pmatrix}-\cos u\\ -\sin u\\ 0\end{pmatrix},\ \frac{d\hat{r}_d(u)}{du}\times\frac{d^2\hat{r}_d(u)}{du^2}=\begin{pmatrix}\sin u\\ -\cos u\\ 1\end{pmatrix},$$

$$\left|\frac{d\hat{r}_d(u)}{du}\times\frac{d^2\hat{r}_d(u)}{du^2}\right|=\sqrt{2},\ \left|\frac{d\hat{r}_d(u)}{du}\right|^2=2,\ \frac{\left|\frac{d\hat{r}_d(u)}{du}\times\frac{d^2\hat{r}_d(u)}{du^2}\right|}{\left|\frac{d\hat{r}_d(u)}{du}\right|^2}=\frac{\sqrt{2}}{2}. \tag{18}$$

The calculation of the required integrals in this case is possible to perform analytically:

$$\int_{u1}^{t}\frac{\left|\frac{d\hat{r}_d(u)}{du}\times\frac{d^2\hat{r}_d(u)}{du^2}\right|}{\left|\frac{d\hat{r}_d(u)}{du}\right|^2}du=\int_{u1}^{t}\frac{\sqrt{2}}{2}du=\frac{\sqrt{2}}{2}(t-u_1),$$

$$\xi=\int_{u_1}^{u}\left[\sqrt{2}\cos\frac{\sqrt{2}}{2}(t-u_1)\right]dt=2\left[\sin\frac{\sqrt{2}}{2}(u-u_1)\right], \tag{19}$$

$$\eta=\int_{u_1}^{u}\left[\sqrt{2}\sin\frac{\sqrt{2}}{2}(t-u_1)\right]dt=2\left[1-\cos\frac{\sqrt{2}}{2}(u-u_1)\right],.$$

and we can finally the obtained equations of the development curve for the edge of regression $\hat{r}_d(u)$ in the parametric form

$$\begin{cases}\xi=2\left[\sin\frac{\sqrt{2}}{2}(u-u_1)\right],\\ \eta=2\left[1-\cos\frac{\sqrt{2}}{2}(u-u_1)\right],\end{cases}\quad u_1\le u\le u_2.. \tag{20}$$

The development curve for the edge of regression, obtained according to equations (20) is shown in Figure 7.

The development of the curve, which lies on a tangent developable surface and is expressed through the edge of regression of this surface, can be obtained using the equations of development curve for the edge of regression. If the line can be written by the equation (11), the equations of its development in the edge of regression development plane will take the form of

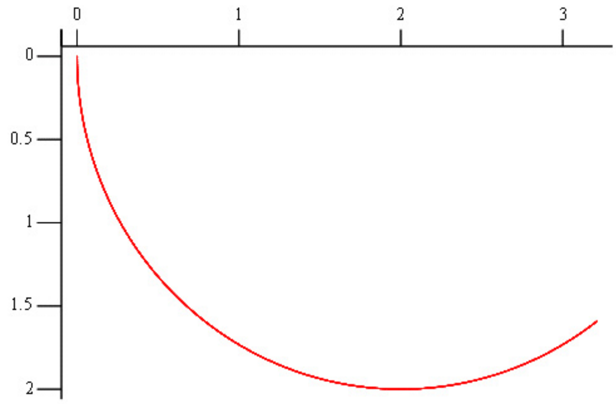

Figure 7. Development curve for edge of regression $\hat{r}_d(u)=\begin{pmatrix}\cos u\\ \sin u\\ u\end{pmatrix}$ with parameter $0\le u\le\pi$

$$\begin{cases}\xi_1=\xi+\left|\hat{r}_d(u)-\hat{r}_1(u)\right|\cdot\cos\alpha,\\ \eta_1=\eta+\left|\hat{r}_d(u)-\hat{r}_1(u)\right|\cdot\sin\alpha,\end{cases}\quad u_1\le u\le u_2, \tag{21}$$

where

$$\cos\alpha=\frac{d\xi/du}{\sqrt{\left(\frac{d\xi}{du}\right)^2+\left(\frac{d\eta}{du}\right)^2}},\ \sin\alpha=\frac{d\eta/du}{\sqrt{\left(\frac{d\xi}{du}\right)^2+\left(\frac{d\eta}{du}\right)^2}}. \tag{22}$$

Using the proposed method it is possible to build the development for the tangent developable surface element defined by the equation (14). We will need to find the equations of the development curve for the directive curve of this surface element, while the equations of the development curve for the edge of regression were found above.

Taking into account the formulas (13) and (2) the guiding curve equation can be written in the following form:

$$\hat{r}_1(u)=\hat{r}_d(u)+\sqrt{2}\hat{\tau}=\begin{pmatrix}\cos u-\sin u\\ \sin u+\cos u\\ u+1\end{pmatrix},\ u_1\le u\le u_2. \tag{23}$$

Then we can get

$$\begin{cases} \xi_1 = \xi + \sqrt{2} \cdot \cos\alpha, \\ \eta_1 = \eta + \sqrt{2} \cdot \sin\alpha. \end{cases}, \ u_1 \le u \le u_2. \tag{24}$$

Let's perform the required calculations:

$$\frac{d\xi}{du} = \sqrt{2}\left[\cos\frac{\sqrt{2}}{2}(u-u_1)\right], \ \frac{d\eta}{du} = \sqrt{2}\left[\sin\frac{\sqrt{2}}{2}(u-u_1)\right],$$

$$\sqrt{\left(\frac{d\xi}{du}\right)^2 + \left(\frac{d\eta}{du}\right)^2} = \sqrt{2\left[\cos\frac{\sqrt{2}}{2}(u-u_1)\right]^2 + 2\left[\sin\frac{\sqrt{2}}{2}(u-u_1)\right]^2} = \sqrt{2}, \tag{25}$$

$$\cos\alpha = \cos\frac{\sqrt{2}}{2}(u-u_1), \ \sin\alpha = \sin\frac{\sqrt{2}}{2}(u-u_1),$$

and finally obtain the equations of the curve $\hat{r}_1(u)$ development in the parametric form:

$$\begin{cases} \xi_1 = 2\left[\sin\frac{\sqrt{2}}{2}(u-u_1)\right] + \sqrt{2} \cdot \cos\frac{\sqrt{2}}{2}(u-u_1), \\ \eta_1 = 2\left[1-\cos\frac{\sqrt{2}}{2}(u-u_1)\right] + \sqrt{2} \cdot \sin\frac{\sqrt{2}}{2}(u-u_1), \end{cases} \quad u_1 \le u \le u_2.. \tag{26}$$

The development of the tangent developable surface element, obtained according the to equations (20) and (26) is shown in Figure 8.

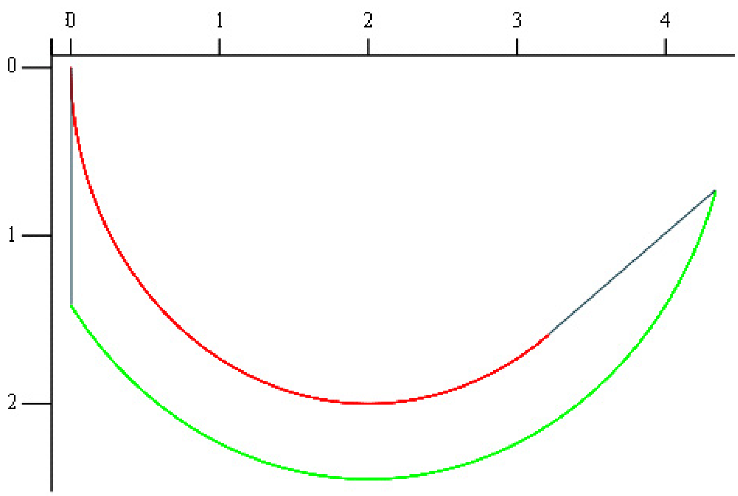

Figure 8. Development of the tangent developable surface element

CONCLUSION

Tangent developable surface along with a conical and cylindrical surfaces provide ample opportunity for solving problems in various fields of technology. Described modeling methods of the tangent developable surfaces and their developments can be applied in the design and manufacturing of plate and tent structures with various materials. The need for the unfolding curve expressing through the edge of regression of the tangent developable surfaces limits the algorithm application ability; therefore, the authors aim is to get developing algorithm for arbitrary tangent developable surface element.

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